Positive Common Priors and Speculative Trade*

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Abstract

The speculative trade theorem specifies that a positive common prior, which assigns positive probability to all elements of the join of the agents' partitions, implies that there can be no mutually beneficial trade that is common knowledge at *some* state. We show that the reverse is also true for full support type structures, where at each state a type assigns positive probability to the element of the join that contains this state. By providing this behavioral characterization of positive common priors, we complement the existing result of the literature, that for arbitrary type structures there is a (not necessarily positive) common prior if and only if there is no mutually beneficial trade that is common knowledge at *all* states.

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1 Introduction

The common prior assumption (Harsanyi [1968]) is prevalent in games with incomplete information. It requires that there exists a probability distribution that generates the posterior beliefs of the types of all agents, by updating on their information using Bayes' rule, whenever possible.[1] This last qualification means that if the common prior assigns zero probability to some of the agents' types, the posteriors of these types

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[1]See Morris [1995] for a critical overview of the common prior assumption in economics and Gul [1998], Aumann [1998] for a discussion on its merits.

can be formulated arbitrarily. A positive common prior assigns positive probability to the join of the agents' partitions, hence not allowing arbitrary posteriors.

The implications of the common prior assumption have been analysed in a variety of settings.[2] In addition, Aumann [1976] shows that a *positive* common prior implies that it cannot be common knowledge at *some* state that posteriors are different. Sebenius and Geanakoplos [1983] and Milgrom and Stokey [1982] extend this result to formulate the speculative trade theorem, which specifies that there does not exist a mutually beneficial trade that is common knowledge at *some* state. Our contribution is to show that the reverse is also true for full support type structures, hence providing a behavioral characterization of positive common priors, in terms of trading behavior. This is important, because in most games a full support type structure is assumed.[3]

Several papers (Morris [1994], Feinberg [2000], Samet [1998a], Bonanno and Nehring [1999], Halpern [2002], Ng [2003]) characterize the existence of a (not necessarily positive) common prior, with respect to the weaker condition that there does not exist a mutually beneficial trade that is common knowledge at *all* states. This means that in order to reject the hypothesis of a common prior, one needs to check that the trade condition applies to the entire state space, rather than just a subset of it, as with the case of a positive common prior. As the following simple example shows, a common prior does not preclude common knowledge trade at some state.

Consider a state space S with three states, s_1, s_2 and s_3. There are two agents, 1 and 2, whose information structure is represented by the same partition $\{(s_1, s_2), (s_3)\}$. Suppose that 1's posterior at s_1 and s_2 on S is $(1/3, 2/3, 0)$, whereas 2's is $(2/3, 1/3, 0)$. At s_3, both have the same posterior, $(0, 0, 1)$. This type structure has one common prior, assigning probability 1 to s_3. Consider a trade that makes agent 1 pay 10 to agent 2 if s_1 occurs, with the opposite payment if s_2 occurs and no payments at s_3. Then, at s_1 and s_2 both agents expect strictly positive gains from this trade, which is mutually beneficial. This fact is common knowledge at s_1 and s_2.

Observe that the events that can be common knowledge are $\{s_1, s_2\}$, $\{s_3\}$ and S. We call each of the first two a smallest public event, because whenever it occurs everyone knows it, and there does not exist a smaller event with this property.[4] An

[2]For an incomplete list, see Mertens and Zamir [1985], Rubinstein and Wolinsky [1990], Aumann and Brandenburger [1995], Samet [1998b], Lipman [2003, 2010], Heifetz [2006], Barelli [2009], Rodrigues-Neto [2009], Hellman [2011], Lehrer and Samet [2011], Lehrer and Samet [2014] and Liu [2015].

[3]In fact, the stronger assumption is usually made, that each type assigns positive probability to the true state.

[4]A smallest public event is an element of the meet of the agents' partitions. The meet is the finest partition of S that is coarser than the partition of each agent. The join is the coarsest partition that is finer than the partition of each agent. In general, smallest public events can describe heterogeneous information across agents, although this was not necessary for the purposes of this example.

event which describes common knowledge trade at a state must include at least one smallest public event. A common prior allows common knowledge of different beliefs and therefore speculative trade in a smallest public event, such as $\{s_1, s_2\}$, as long as it is not positive, so that it assigns zero probability on that event. In that case, posterior beliefs can be formulated arbitrarily.

The contribution of this paper is that, for full support type structures, no positive common prior implies common knowledge trade at some state. To show this, we establish that for each smallest public event E, there is a common prior with support on E if and only if there is a common prior on E that assigns positive probability to each element of the join of the agents' partitions that is contained in E.

Samet [1998b] shows that it is without loss of generality to characterize the existence of a common prior in a minimal state space S, where the only smallest public event is S. He argues this by showing that, for each smallest public event E there exists at most one common prior with support on E and any common prior on S is in the convex hull of these common priors. This implies that there exists a common prior if and only if there is a common prior with support on some smallest public event. However, with positive common priors this is not the case, because a positive common prior on E does not imply the existence of a positive common prior on S, as shown in the example. In particular, a positive common prior is in the interior of the convex hull of the positive common priors for each E.

The paper proceeds as follows. In Section 2 we define the type structure and the notion of the positive common prior. In Section 3 we provide the characterization of no positive common priors in terms of speculative trade at some state.

2 Type structures and positive common priors

Consider a state space S with finitely many elements and set I of finitely many agents. Agent i's information structure is represented by possibility correspondence $P^i : S \to 2^S \setminus \emptyset$. At $s \in S$, agent i considers $P^i(s) \subseteq S$ to be possible. We assume that P^i generates a partition of S, so that $s \in P^i(s)$ and $s' \in P^i(s)$ implies $P^i(s') = P^i(s)$. We refer to P^i both as the possibility correspondence and the partition it generates. If $P^i(s) \subseteq P^j(s)$ for each $s \in S$, we say that P^i is finer than P^j and P^j is coarser than P^i. The meet P of the agents' partitions consists of the finest partition of S that is coarser than each P^i. We call each $E \in P$ a smallest public event.

An event is a subset of S. We say that i knows event E at $s \in S$ if in all states he considers possible, E is true, so that $P^i(s) \subseteq E$. We say that event E is common knowledge at $s \in S$ if $s \in E' \subseteq E$, where E' is an element of the meet P.

A type structure $\{t^i, P^i\}_{i \in I}$ consists of a possibility correspondence P^i and a function $t^i : S \times 2^S \to [0,1]$, where $t^i(s, E)$ is agent i's posterior at $s \in S$ about event $E \subseteq S$. For each $s \in S$, we assume that $t^i(s, P^i(s)) = 1$ and $t^i(s, E) = \sum_{s' \in E} t^i(s, s')$ for all $E \subseteq S$. We say that $\{t^i, P^i\}_{i \in I}$ is a *full support* type structure if $t^i(s, \bigcap_{j \in I} P^j(s)) > 0$, where $\bigcap_{j \in I} P^j(s)$ is the join of the agents' partitions at s.[5] Moreover, $s' \in P^i(s)$ implies $t^i(s, \cdot) = t^i(s', \cdot)$. Given prior $\pi \in \Delta S$ and events E, F with $\pi(F) > 0$, let $\pi(E|F) = \frac{\pi(E \cap F)}{\pi(F)}$. We next define the notion of a positive common prior.

Definition 1. *Prior $\pi \in \Delta S$ is a prior for i if, for each $s \in S$ with $\pi(P^i(s)) > 0$, we have $t^i(s, E) = \pi(E|P^i(s))$ for each $E \subseteq S$. It is a positive prior for i if, additionally, $\pi(P^i(s)) > 0$ for each $s \in S$. Type structure $\{t^i, P^i\}_{i \in I}$ has a (positive) common prior if there exists $\pi \in \Delta S$ that is a (positive) prior for each $i \in I$.*

Note that Aumann [1976] and Sebenius and Geanakoplos [1983] define a positive common prior such that $\pi(\bigcap_{j \in I} P^j(s)) > 0$, instead of the weaker condition that $\pi(P^j(s)) > 0$ for all $j \in I$. However, given our assumption that $t^i(s, \bigcap_{j \in I} P^j(s)) > 0$ for each $i \in I$, these two definitions are equivalent.

3 Common knowledge trade

A trade $b = \{b^i\}_{i \in I}$ is a collection of functions $b^i : S \to \mathbb{R}$, such that $\sum_{i \in I} b^i(s) = 0$ for each $s \in S$. At $s \in S$, agent i's payoff from b is $\sum_{s' \in S} t^i(s, s') b^i(s') > 0$. Let $B^b = \{s \in S : \sum_{s' \in S} t^i(s, s') b^i(s') > 0, \forall i \in I\}$ be the event that all agents expect strictly positive gains from trade b.

Samet [1998a] shows that for arbitrary type spaces, there is no common prior if and only if there exists trade b such that $B^b = S$. Hence, the trade is common knowledge at *all* states. The following theorem shows that for full support type spaces, there is no positive common prior if and only if there exists trade b such that B^b is common knowledge at *some* state $s \in S$. In other words, speculative trade at some state is possible if and only if there is no positive common prior.

To provide a sketch of the proof of one direction, suppose that there is no positive common prior. Say that a common prior is positive given E if it assigns positive

[5] Whereas Samet [1998b] makes the stronger assumption that $t^i(s, s) > 0$, Samet [1998a] does not assume $t^i(s, \bigcap_{j \in I} P^j(s)) > 0$.

probability to each element of the join of the agents' partitions which is contained in E. If, for each smallest public event E, there is a positive common prior given E, then by taking a convex combination of them with strictly positive weights we can construct a positive common prior with support on S, which is a contradiction. Hence, for some smallest public event E, there is no positive common prior given E. If there is also no common prior with support on E, we can apply the characterization result of the common prior by Samet [1998a], restricted on E, to show that there is common knowledge trade at all states in E.

The only remaining case is that there exists a common prior with support on E which is not positive given E. This means that it assigns zero probability to some element A of the join of the agents' partition, which is contained in E. Because it has support on E, it must also assign positive probability to some other element B of the join which is contained in E. Because E is a smallest public event, we argue in the proof that, starting from B and an agent's partition element which intersects B and has positive prior probability, we can "reach" A, using a sequence of agents whose partition elements also have positive prior probability, thus showing that also A must have positive prior probability and reaching a contradiction. Hence, the key insight is that for full support type spaces, a common prior with support on a smallest public event is equivalent to a positive common prior given that event.

Theorem 1. *Full support type structure $\{t^i, P^i\}_{i\in I}$ has a positive common prior if and only if there is no trade b such that event B^b is common knowledge at some $s \in S$.*

Proof. Fix type structure $\{t^i, P^i\}_{i\in I}$ and suppose there exist trade b and state $s \in S$ such that B^b is common knowledge at s. We need to show that $\{t^i, P^i\}_{i\in I}$ has no positive common prior.

Suppose that $\{t^i, P^i\}_{i\in I}$ has a positive common prior π. There is a smallest public event E such that $s \subseteq E \subseteq B^b$. Because $E \subseteq B^b$ and E is partitioned by each P^i, we have that $\sum_{s'\in E} t^i(s_1, s')b^i(s') > 0$ for all $s_1 \in E$, $i \in I$. Because π is a positive prior for i, $\pi(P^i(s_1)) > 0$, hence $t^i(s_1, s') = \frac{\pi(s')}{\pi(P^i(s_1))}$ for all $s_1 \in E$, which implies that $\sum_{s'\in E} \pi(s')b^i(s') > 0$. By adding over all agents we have $\sum_{i\in I}\sum_{s'\in E} \pi(s')b^i(s') = \sum_{s'\in E} \pi(s')\sum_{i\in I} b^i(s') > 0$. Since $\sum_{i\in I} b^i(s') = 0$ for all $s' \in E$, we have a contradiction.

Conversely, suppose that $\{t^i, P^i\}_{i\in I}$ has no positive common prior. We need to show that there exist state $s \in S$ and trade b such that B^b is common knowledge at s.

Say that event E is a smallest public event if there does not exist public event E' such that $E' \subseteq E$. Let $\mathcal{E}$ be the collection of all smallest public events, with cardinality $|\mathcal{E}|$. Suppose first that for each $E \in \mathcal{E}$, there is a common prior π_E, with support on

E, such that $\pi_E(P^i(s)) > 0$ for each $i \in I$ and $s \in E$. Because $\mathcal{E}$ is a partition of S, $\pi = \sum_{E \in \mathcal{E}} \frac{1}{|\mathcal{E}|}\pi_E$ is a positive common prior, a contradiction. Hence, for some $E \in \mathcal{E}$, there is no common prior, with support on E, such that $\pi_E(P^i(s)) > 0$ for each $i \in I$ and $s \in E$. We next show that there is no common prior with support on E.

To prove by contradiction, suppose there is a common prior π_E, with support on E, such that $\pi_E(P^i(s')) = 0$ for some $i \in I$ and $s' \in E$. Because π_E has support on E, $\pi_E(P^j(s'')) > 0$ for some $s'' \in E$ and $j \in I$. Moreover, because E is a smallest public event, we can "reach" s' starting from s'', using a sequence of agents $i_1, i_2, \ldots, i_n$ and states $s_1, s_2, \ldots, s_n \in E$, where $i_1 = j$, $i_n = i$, $s_1 = s''$, $s_n = s'$, such that $P^{i_k}(s_k) \cap P^{i_{k+1}}(s_{k+1}) \neq \emptyset$, $k = 1, \ldots, n-1$.[6] Let $s_0 \in P^{i_1}(s_1) \cap P^{i_2}(s_2) \neq \emptyset$, which implies $t^{i_1}(s_1, \cdot) = t^{i_1}(s_0, \cdot)$, $P^{i_1}(s_1) = P^{i_1}(s_0)$ and $P^{i_2}(s_2) = P^{i_2}(s_0)$. Because $\pi_E(P^{i_1}(s_1)) = \pi_E(P^{i_1}(s_0)) > 0$ and $t^i(s_0, \bigcap_{j \in I} P^j(s_0)) > 0$, we have $\pi_E(\bigcap_{j \in I} P^j(s_0)) > 0$, which implies $\pi_E(P^{i_2}(s_0)) = \pi_E(P^{i_2}(s_2)) > 0$. Continuing inductively, we have that $\pi_E(P^{i_n}(s_n)) > 0$, where $i_n = i$, $s_n = s'$, a contradiction.

The Corollary in Samet [1998a] shows that there is no common prior if and only if $B^b = S$ for some trade b. Applied in our setting, no common prior with support on E is equivalent to the existence of a trade b (that assigns 0 to all agents for states in $S \setminus E$) such that $B^b = E$. Because E is a nonempty public event, it is common knowledge at $s \in B^b = E$.

□

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[6] See Aumann [1976] for this argument of reachability.

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